The strong no loop conjecture is true for mild algebras

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Abstract

Let Λ be a finite dimensional associative algebra over an algebraically closed field with a simple module S of finite projective dimension. The strong no loop conjecture says that this implies $\mathrm{Ext}^1_\Lambda(S,S)=0$, i.e. that the quiver of Λ has no loops in the point corresponding to S. In this paper we prove the conjecture in case Λ is mild, which means that Λ has only finitely many two-sided ideals and each proper factor algebra Λ/J is representation finite. In fact, it is sufficient that a "small neighborhood" of the support of the projective cover of S is mild.

1 Introduction

Let Λ be a finite dimensional associative algebra over a fixed algebraically closed field $\mathbf{k}$ of arbitrary characteristic. We consider only Λ-right modules of finite dimension.

The strong no loop conjecture says that a simple Λ-module S of finite projective dimension satisfies $\mathrm{Ext}^1_\Lambda(S,S)=0$. To prove this conjecture for a given algebra we can switch to the Morita-equivalent basic algebra and therefore assume that $\Lambda=\mathbf{k}\,\mathcal{Q}/I$ for some quiver $\mathcal{Q}$ and some ideal I generated by linear combinations of paths of length at least two. Then $S=S_x$ is the simple corresponding to a point x in $\mathcal{Q}$ and the conjecture means that there is no loop at x provided the projective dimension $\mathrm{pdim}_\Lambda S_x$ is finite.

The conjecture is known for

- monomial algebras by Igusa [Igu90],
- truncated extensions of semi-simple rings by Marmaridis, Papistas [MP95],
- bound quiver algebras $k\,\mathcal{Q}/I$ such that for each loop $\alpha\in\mathcal{Q}$ there exists an $n\in\mathbb{N}$ with $\alpha^n\in I\setminus(IJ+JI)$, where J denotes the ideal generated by the arrows [GSZ01],
- special biserial algebras by Liu, Morin [LM04],
- two point algebras with radical cube zero by Jensen [Jen05].

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In this paper, we prove the conjecture for another class of algebras including all representation-finite algebras. To state our result precisely we introduce for any point x in $\mathcal{Q}$ its **neighborhood** $\Lambda(x) = e\,\Lambda\,e$. Here e is the sum of all primitive idempotents $e_z \in \Lambda$ such that z belongs to the support of the projective $P_x := e_x\,\Lambda$ or such that there is an arrow $z \to x$ in $\mathcal{Q}$ or a configuration $y' \leftarrow x \rightleftarrows y \leftarrow z$ with 4 different points x, y, y' and z.
Recall that an algebra Λ is called **distributive** if it has a distributive lattice of two-sided ideals and **mild** if it is distributive and any proper quotient $\Lambda\,/J$ is representation-finite.

Our main result reads as follows:

Theorem 1.1
Let $\Lambda = \mathbf{k}\,\mathcal{Q}\,/I$ be a finite dimensional algebra over an algebraically closed field $\mathbf{k}$. Let x be a point in $\mathcal{Q}$ such that the corresponding simple Λ-module S_x has finite projective dimension. If $\Lambda(x)$ is mild, then there is no loop at x.

Of course, it follows immediately that the strong no loop conjecture holds for all mild algebras, in particular for all representation-finite algebras.

Corollary 1.2
Let Λ be a mild algebra over an algebraically closed field. Let S be a simple Λ-module. If the projective dimension of S is finite, then $\operatorname{Ext}^1_\Lambda(S,S) = 0$.

In order to prove the theorem we do not look at projective resolutions. Instead we refine a little bit the K-theoretic arguments of Lenzing [Len69, Satz 5], also used by Igusa in his proof of the strong no loop conjecture for monomial algebras [Igu90, Corollary 6.2], to obtain the following result:

Proposition 1.3
Let $\Lambda = \mathbf{k}\,\mathcal{Q}\,/I$ be a finite dimensional algebra, x a point in $\mathcal{Q}$ and α an oriented cycle at x. If P_x has an α-filtration of finite projective dimension, then α is not a loop.

Here an **α-filtration** $\mathcal{F}$ of P_x is a filtration

$$P_x = M_0 \supset M_1 \supset \ldots \supset M_n = 0$$

by submodules with

$$\alpha M_i \subset M_{i+1} \;\forall\; i = 0 \ldots n-1.$$

The filtration $\mathcal{F}$ has finite projective dimension if $\operatorname{pdim}_\Lambda M_i < \infty$ holds for all $i = 1 \ldots n-1$.

This proposition is shown by Lenzing in [Len69, Satz 5] for the special filtration $M_i = \alpha^i\,\Lambda$, but his proof remains valid for all α-filtrations.

Our strategy to prove Theorem 1.1 is then as follows: We consider the point x with $\operatorname{pdim}_\Lambda S_x < \infty$ and its mild neighborhood $A := \Lambda(x)$. We assume in addition that there is a loop α in x. Then we deduce a contradiction either by showing that $\operatorname{pdim}_\Lambda S_x = \infty$ or by constructing a certain α-filtration $\mathcal{F}$ of P_x having finite projective dimension in mod-$\,\Lambda$ and implying that α is not a loop by Proposition 1.3. Since $\Lambda(x)$ contains the support of P_x, this filtrations coincide for P_x as a Λ-module and as a $\Lambda(x)$-module. Thus we are dealing with a mild algebra, and we use in an essential way the deep structure theorems about such algebras given in [BGRS85] and [Bon09] to obtain the wanted α-filtrations. In particular, we show that we always work in the ray-category attached to $\Lambda(x)$. This makes it much easier to use cleaving diagrams. But still the construction of the appropriate α-filtrations depends on the study of several cases and it remains a difficult technical problem. The α-filtrations are always built in such a way that they have finite projective dimension in mod-$\,\Lambda$ provided $\operatorname{pdim}_\Lambda S_x < \infty$.

To illustrate the method by two examples we define $\langle w_1, \ldots, w_k \rangle$ as the submodule of P_x generated by elements $w_1, \ldots, w_k \in P_x$.

Example 1.4
Let Λ be an algebra such that $\Lambda(x)$ is given by the quiver

$$\mathcal{Q} = \begin{tikzcd} & x \arrow[loop above, "\alpha"] \arrow[dl, "\gamma_1"'] \arrow[d, "\beta_1"] & \\ z \arrow[r, "\gamma_2"] & y_1 \arrow[r, "\beta_2"] & y_2 \arrow[ul, "\beta_3"'] \end{tikzcd}$$

and a relation ideal I such that the projective module P_x is described by the following graph:

$$\begin{tikzcd} & e_x \arrow[dl] \arrow[d] \arrow[dr] & \\ \gamma_1 \arrow[d, "\gamma_2"'] & \alpha \arrow[dl] \arrow[d] & \beta_1 \arrow[d] \\ \alpha\beta_1 & \alpha^2 & \beta_1\beta_2 \arrow[l] \end{tikzcd} .$$

Notice that the picture means that there are relations $\alpha^2 - \lambda_1\, \beta_1\beta_2\beta_3,\ \alpha\beta_1 - \lambda_2\, \gamma_1\gamma_2 \in I$ for some $\lambda_i \in \mathbf{k}\setminus\{0\}$. From the obvious exact sequences

$$0 \to \operatorname{rad} P_x \to P_x \to S_x \to 0$$

$$0 \to \langle \beta_1, \gamma_1 \rangle \to \operatorname{rad} P_x \to S_x \to 0$$
$$0 \to \langle \alpha^2, \gamma_1 \rangle \to \langle \alpha, \gamma_1 \rangle \to S_x \to 0$$

we see that $\operatorname{pdim}_\Lambda S_x < \infty$ leads to $\operatorname{pdim}_\Lambda \operatorname{rad} P_x < \infty$ and $\operatorname{pdim}_\Lambda \langle \beta_1, \gamma_1 \rangle < \infty$. Since $\langle \beta_1, \gamma_1 \rangle = \langle \beta_1 \rangle \oplus \langle \gamma_1 \rangle$ and $\langle \alpha^2, \gamma_1 \rangle = \langle \alpha^2 \rangle \oplus \langle \gamma_1 \rangle$ in this example, both $\operatorname{pdim}_\Lambda \langle \gamma_1 \rangle$ and $\operatorname{pdim}_\Lambda \langle \alpha, \gamma_1 \rangle$ are finite. Then the following α-filtration $\mathcal{F}$: $P_x \supset \langle \alpha, \gamma_1 \rangle \supset \langle \alpha^2 \rangle \supset 0$ has finite projective dimension in mod-Λ.

In the next example we see that this method may not work if the neighborhood $\Lambda(x)$ is not mild, even if the support of P_x is mild.

Example 1.5
Let $\Lambda(x) = \mathbf{k}\, \mathcal{Q}/I$ be given by the quiver

$$\mathcal{Q} = \begin{tikzcd} x \arrow[loop left, "\alpha"] \arrow[r, bend left, "\beta_1"] \arrow[d, "\gamma"] & y \arrow[l, bend left, "\beta_2"] \arrow[d, bend left] \\ z \arrow[loop left, "\delta"] \arrow[r, bend left] & z' \arrow[l, bend left] \arrow[u] \end{tikzcd}$$

and by a relation ideal I such that P_x is represented by

$$\begin{tikzcd} & e_x \arrow[dl] \arrow[d] \arrow[dr] & \\ \gamma \arrow[d, "\delta"'] & \alpha \arrow[dl] \arrow[d] & \beta_1 \arrow[dl, "\beta_2"] \\ \alpha\gamma & \alpha^2 & \end{tikzcd} .$$

Here we get stuck because the uniserial module with basis $\{\gamma, \alpha\gamma\}$ allows only the composition series as an α-filtration. Since we do not know $\operatorname{pdim}_\Lambda S_z$, which depends on Λ and not only on $\Lambda(x)$, our method does not apply.

The article is organized as follows: In the second section we recall some facts about ray-categories and we show how to reduce the proof to standard algebras without penny-farthings. This case is then analyzed in the last section.

The results of this article are contained in my PhD-thesis written at the University of Wuppertal.

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2 The reduction to standard algebras

2.1 Ray-categories and standard algebras

We recall some well-known facts from [BGRS85], [GR92].

Let $A := \Lambda(x) = \mathbf{k}\,\mathcal{Q}_A\,/I_A$ be a basic distributive $\mathbf{k}$-algebra. Then every space e_xAe_y is a cyclic module over e_xAe_x or e_yAe_y and we can associate to A its **ray-category** $\overrightarrow{A}$. Its objects are the points of $\mathcal{Q}_A$. The morphisms in $\overrightarrow{A}$ are called **rays** and $\overrightarrow{A}(x,y)$ consists of the orbits $\overrightarrow{\mu}$ in e_xAe_y under the obvious action of the groups of units in e_xAe_x and e_yAe_y. The composition of two morphisms $\overrightarrow{\mu}$ and $\overrightarrow{\nu}$ is either the orbit of the composition $\mu\nu$, in case this is independent of the choice of representatives in $\overrightarrow{\mu}$ and $\overrightarrow{\nu}$, or else 0. We call a non-zero morphism $\eta \in \overrightarrow{A}$ **long** if it is non-irreducible and satisfies $\nu\eta = 0 = \eta\nu'$ for all non-isomorphisms $\nu, \nu' \in \overrightarrow{A}$. One crucial fact about ray-categories frequently used in this paper is that A is mild iff $\overrightarrow{A}$ is so [GR92, see Theorem 13.17].

The ray-category is a finite category characterized by some nice properties. For instance, given $\lambda\,\mu\kappa = \lambda\,\nu\kappa \neq 0$ in $\overrightarrow{A}$, $\mu = \nu$ holds. We shall refer to this property as the **cancellation law**.

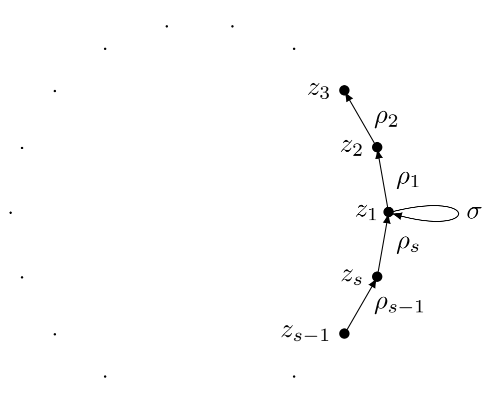
Given $\overrightarrow{A}$, we construct in a natural way its linearization $\mathbf{k}(\overrightarrow{A})$ and obtain a finite dimensional algebra

$$\overline{A} = \bigoplus_{x,y\in\mathcal{Q}_A} \mathbf{k}(\overrightarrow{A})(x,y),$$

the **standard form** of A. In general, A and $\overline{A}$ are not isomorphic, but they are if either A is minimal representation-infinite [Bon09, Theorem 2] or representation-finite with $\operatorname{char}\mathbf{k} \neq 2$ [GR92, Theorem 13.17].

Similar to A, the ray-category $\overrightarrow{A}$ admits a description by quiver and relations. Namely, there is a canonical full functor $\vec{\ } : \mathcal{P}\,\mathcal{Q}_A \to \overrightarrow{A}$ from the path category of $\mathcal{Q}_A$ to $\overrightarrow{A}$. Two paths in $\mathcal{Q}_A$ are **interlaced** if they belong to the transitive closure of the relation given by $v \sim w$ iff $v = pv'q,\ w = pw'q$ and $\overrightarrow{v'} = \overrightarrow{w'} \neq 0$, where p and q are not both identities.

A **contour** of $\overrightarrow{A}$ is a pair (v, w) of non-interlaced paths with $\overrightarrow{v} = \overrightarrow{w} \neq 0$. Note that these contours are called essential contours in [BGRS85, 2.7]. Throughout this paper we will need a special kind of contours called penny farthings. A **penny-farthing** P in $\overrightarrow{A}$ is a contour $(\sigma^2, \rho_1 \ldots \rho_s)$ such that the full subquiver $\mathcal{Q}_P$ of $\mathcal{Q}_A$ that supports the arrows of P has the following shape:



Moreover, we ask the full subcategory $A_P \subset A$ living on $\mathcal{Q}_P$ to be defined by $\mathcal{Q}_P$ and one of the following two systems of relations

$$0 = \sigma^2 - \rho_1 \dots \rho_s \quad = \quad \rho_s\rho_1 = \rho_{i+1} \dots \rho_s \sigma \rho_1 \dots \rho_{f(i)}, \tag{1}$$

$$0 = \sigma^2 - \rho_1 \dots \rho_s \quad = \quad \rho_s\rho_1 - \rho_s\sigma\rho_1 = \rho_{i+1} \dots \rho_s \sigma \rho_1 \dots \rho_{f(i)}, \tag{2}$$

where $f : \{1, 2, \dots, s-1\} \to \{1, 2, \dots, s\}$ is some non-decreasing function (see [BGRS85, 2.7]. For penny-farthings of type (1) A_P is standard, for that of type (2) A_P is not standard in case the characteristic is two.

A functor $F : D \to \overrightarrow{A}$ between ray categories is **cleaving** ([GR92, 13.8]) iff it satisfies the following two conditions and their duals:

a) $F(\mu) = 0$ iff $\mu = 0$.

b) If $\eta \in D(y, z)$ is irreducible and $F(\mu) : F(y) \to F(z')$ factors through $F(\eta)$ then μ factors already through η.

The key fact about cleaving functors is that $\overrightarrow{A}$ is not representation finite if D is not. In this article D will always be given by its quiver $\mathcal{Q}_D$, that has no oriented cycles and some relations. Two paths between the same points give always the same morphism, and zero relations are indicated by a dotted line. As in [GR92, section 13], the cleaving functor is then defined by drawing the quiver of D with relations and by writing the morphism $F(\mu)$ in $\overrightarrow{A}$ close to each arrow μ.

By abuse of notation, we denote the irreducible rays of $\overrightarrow{A}$ and the corresponding arrows of $\mathcal{Q}_A$ by the same letter.

2.2 Getting rid of penny-farthings

Using the above notations let $P = (\sigma^2, \rho_1 \dots \rho_s)$ be a penny-farthing in $\overrightarrow{A}$. We shall show now that $x = z_1$. Therefore $\sigma = \alpha$ and P is the only penny-farthing in $\overrightarrow{A}$ by [GR92, Theorem 13.12].

Lemma 2.1
If there is a penny-farthing $P = (\sigma^2, \rho_1 \dots \rho_s)$ in $\overrightarrow{A}$, then $z_1 = x$.

Proof. We consider two cases:

- $x \in \mathcal{Q}_P$: Hence $\mathcal{Q}_P$ has the following shape:

$$\begin{tikzcd} z_1 \arrow[loop left, "\sigma"] \arrow[rr, bend left, "\rho_1 \cdots \rho_l"] & & x \arrow[loop right, "\alpha"] \arrow[ll, bend left, "\rho_{l+1} \cdots \rho_s"] \end{tikzcd}$$

  But this can be the quiver of a penny-farthing only for $z_1 = x$.

- $x \notin \mathcal{Q}_P$: Since A is the neighborhood of x, only the following cases are possible:

  a) $e_x A e_z \neq 0$: Since $x \notin \mathcal{Q}_P$ we can apply the dual of [Bon85, Theorem 1] or [GR92, Lemma 13.15] to $\overrightarrow{A}$ and we see that the following quivers occur as subquivers of $\mathcal{Q}_A$:

$$\begin{tikzcd} z_1 \arrow[loop left, "\sigma"] \arrow[r, bend left, "\rho_1"] & z_2 \arrow[l, bend left, "\rho_2"] \\ x \arrow[loop left, "\alpha"] \arrow[u] & \end{tikzcd} \quad resp. \quad \begin{tikzcd} z_1 \arrow[loop left, "\sigma"] \arrow[r, bend left, "\rho_1"] & z_2 \arrow[l, bend left, "\rho_2"] \\ & x \arrow[loop left, "\alpha"] \arrow[u] \end{tikzcd} .$$

Moreover, there can be only one arrow starting in x. This is a contradiction to the actual setting.

b) $\exists\, z_1 \to x$: By applying [Bon85, Theorem 1] or the dual of [GR92, Lemma 13.15] we deduce that the following quiver occurs as a subquiver of $\mathcal{Q}_A$:

$$\begin{tikzcd}
z_1 \arrow[loop left, "\sigma"] \arrow[r, bend left, "\rho_1"] \arrow[d] & z_2 \arrow[l, bend left, "\rho_2"] \\
x \arrow[loop left, "\alpha"] &
\end{tikzcd}$$

and there can be only one arrow ending in x contradicting the present case.

c) $\exists\, y' \leftarrow x \rightleftarrows y \leftarrow z_1$: If $y \notin \mathcal{Q}_P$, then

$$\begin{tikzcd}
z_1 \arrow[loop left, "\sigma"] \arrow[r, bend left, "\rho_1"] \arrow[d] & z_2 \arrow[l, bend left, "\rho_2"] \\
y \arrow[r, bend left] & x \arrow[l, bend left]
\end{tikzcd}$$

is a subquiver of $\mathcal{Q}_A$ leading to the same contradiction as in b).
If $y \in \mathcal{Q}_P$, then $y = z_2$ and the quiver

$$\begin{tikzcd}
x \arrow[loop left, "\alpha"] \arrow[r, bend left, "\beta_1"] \arrow[d, "\gamma"'] & z_2 \arrow[l, bend left, "\beta_2"] & z_1 \arrow[l, "\rho_1"'] \arrow[loop right, "\sigma"] \\
y' & &
\end{tikzcd}$$

is a subquiver of $\mathcal{Q}_A$. Since $x \notin \mathcal{Q}_P$, all morphisms occurring in the following diagram

$$D := \begin{tikzcd}
\bullet & \bullet \arrow[l, "\rho_2"'] \arrow[r, "\beta_2"] & \bullet & \bullet \arrow[l, "\alpha"'] \arrow[r, "\beta_1"] \arrow[d, "\gamma"'] & \bullet & \bullet \arrow[l, "\rho_1"'] \arrow[r, "\sigma"] & \bullet \\
& & & \bullet & & &
\end{tikzcd}$$

are irreducible and pairwise distinct. Therefore D is a cleaving diagram in $\overrightarrow{A}$. Moreover, some long morphism $\eta = \nu\sigma^3\nu'$ does not occur in D; hence D is still cleaving in $\overrightarrow{A}/\eta$ by [Bon09, Lemma 3]. Since D is of representation-infinite Euclidean type $\widetilde{E}_7$, $\overrightarrow{A}/\eta$ is representation-infinite contradicting the mildness of A.

$\square$

Now, we show that, provided the existence of a penny-farthing in $\overrightarrow{A}$, there exists an α-filtration of P_x having finite projective dimension.

Lemma 2.2
Let $A = \Lambda(x)$ be mild and standard. If there is a penny-farthing in $\overrightarrow{A}$, then there exists an α-filtration $\mathcal{F}$ of P_x having finite projective dimension.

Proof. If there is a penny-farthing P in $\overrightarrow{A}$, then $P = (\alpha^2, \rho_1 \dots \rho_s)$ is the only penny-farthing in $\overrightarrow{A}$ by the last lemma. Since A is standard and mild, there are three cases for the graph of P_x which can occur by [Bon85, Theorem 1] or the dual of [GR92, Lemma 13.15].

I) There exists an arrow $\gamma : x \to z,\ \gamma \neq \rho_1$. Then $s = 2$, the quiver

$$\begin{tikzcd} x \arrow[loop left, "\alpha"] \arrow[r, bend left, "\rho_1"] \arrow[d, "\gamma"] & y \arrow[l, bend left, "\rho_2"] \\ z & \end{tikzcd}$$

is a subquiver of $\mathcal{Q}_A$, and P_x is represented by the following graph:

$$\begin{tikzcd} & e_x \arrow[dl] \arrow[d] \arrow[dr] & \\ \gamma & \alpha \arrow[d] \arrow[dr] & \rho_1 \arrow[dl] \\ & \alpha^2 \arrow[d] & \alpha\rho_1 \arrow[dl] \\ & \alpha^3 & \end{tikzcd} .$$

Let M be a quotient of P_x defined by the following exact sequence:

$$0 \to \langle \gamma \rangle \oplus \langle \rho_1, \alpha\rho_1 \rangle \to P_x \to M \to 0.$$

Then M has S_x as the only composition factor. Hence $\operatorname{pdim}_\Lambda M < \infty$ and $\operatorname{pdim}_\Lambda \langle \rho_1, \alpha\rho_1 \rangle < \infty$. Now, we consider the exact sequence

$$0 \to \langle \alpha^3 \rangle \to \langle \rho_1, \alpha\rho_1 \rangle \to \langle \rho_1 \rangle / \langle \alpha^3 \rangle \oplus \langle \alpha\rho_1 \rangle / \langle \alpha^3 \rangle \to 0.$$

But $\langle \alpha^3 \rangle \cong S_x$ and $\operatorname{pdim}_\Lambda S_x < \infty$, hence $\langle \alpha\rho_1 \rangle / \langle \alpha^3 \rangle \cong S_y$ has finite projective dimension in mod-Λ. Finally, the α-filtration $P_x \supset \langle \alpha \rangle \supset \langle \alpha^2 \rangle \supset \langle \alpha^3 \rangle \supset 0$ has finite projective dimension since all filtration modules $\neq P_x$ have S_x and S_y as the only composition factors.

II) In the second case there exists a point $z \notin \mathcal{Q}_P$ such that $A(x, z) \neq 0$. Then $s = 2$, the quiver

$$\begin{tikzcd} x \arrow[loop left, "\alpha"] \arrow[r, bend left, "\rho_1"] & y \arrow[l, bend left, "\rho_2"] \arrow[d, "\delta"] \\ & z \end{tikzcd}$$

is a subquiver of $\mathcal{Q}_A$, and P_x is represented by:

$$\begin{tikzcd}
e_x \arrow[d] \arrow[dr] & & \\
\alpha \arrow[d] \arrow[dr] & \rho_1 \arrow[dl] \arrow[dr] & \\
\alpha^2 \arrow[d] & \alpha\rho_1 \arrow[dl] & \rho_1\delta \\
\alpha^3 & &
\end{tikzcd} .$$

With similar considerations as in I) we obtain that the same filtration fits.

III) In the last possible case we have $A(x,z) = 0$ for all points $z \notin \mathcal{Q}_P$. Hence P_x is represented by:

$$\begin{tikzcd}
 & e_x \arrow[r] \arrow[d] & \rho_1 \arrow[d] \\
\alpha\rho_1 \arrow[d] & \alpha \arrow[l] \arrow[d] & \rho_1\rho_2 \arrow[d, squiggly] \\
\alpha\rho_1\rho_2 \arrow[d, squiggly] & \alpha^2 \arrow[d] & \rho_1\rho_2\cdots\rho_{s-1} \arrow[l] \\
\alpha\rho_1\rho_2\cdots\rho_{s-1} \arrow[r] & \alpha^3 &
\end{tikzcd} .$$

As a Λ-Module, $M := P_x/\langle\alpha^2\rangle$ has finite projective dimension since $\langle\alpha^2\rangle$ has S_x as the only composition factor. Let K be the kernel of the epimorphism $M \to \langle\alpha^2\rangle$, $e_x \mapsto \alpha^2$, then $K = \langle\rho_1\rangle/\langle\alpha^2\rangle \oplus \langle\alpha\rho_1\rangle/\langle\alpha^3\rangle$ has finite projective dimension. Moreover, $\mathrm{pdim}_\Lambda\langle\rho_1\rangle, \mathrm{pdim}_\Lambda\langle\alpha\rho_1\rangle < \infty$. Since

$$0 \to \langle\alpha\rho_1\rangle \to \langle\alpha\rangle \stackrel{\lambda_\alpha}{\to} \langle\alpha^2\rangle \to 0$$

is exact, $\mathrm{pdim}_\Lambda\langle\alpha\rangle < \infty$. Thus the same filtration as in the first two cases fits again.

□

Lemma 2.3
With above notations let $A = \Lambda(x)$ be mild and non-standard. There exists an α-filtration $\mathcal{F}$ of P_x having finite projective dimension.

Proof. If A is non-standard, then A is representation finite by [Bon09], $\mathrm{char}\,\mathbf{k} = 2$ and there is a penny-farthing in $\overrightarrow{A}$ by [GR92, Theorem 13.17]. Since Lemma 2.1 remains valid, the penny-farthing $(\alpha^2, \rho_1 \ldots \rho_s)$, $\rho_i : z_i \to z_{i+1}$, $z_1 = z_{s+1} = x$, is unique. By [GR92, 13.14, 13.17] the difference between A and $\overline{A}$ in the composition of the arrows shows up in the graphs of the projectives to $z_2, \ldots, z_s$ only. Thus the graph of P_x remains the same in all three cases of the proof of Lemma 2.2 and the filtrations constructed there still do the job. □

3 The proof for standard algebras without penny-farthings

3.1 Some preliminaries

If there is no penny-farthing in $\overrightarrow{A}$, then $A = \overline{A}$ is standard by Gabriel, Roiter [GR92, Theorem 13.17] and Bongartz [Bon09, Theorem 2]. By a result of Liu, Morin [LM04, Corollary 1.3], deduced from a

proposition of Green, Solberg, Zacharia [GSZ01], a power of α is a summand of a polynomial relation in $I = I_\Lambda$. Otherwise $\operatorname{pdim}_\Lambda S_x$ would be infinite contradicting the choice of x. Furthermore, α is a summand of a polynomial relation in I_A by definition of A. But I_A is generated by paths and differences of paths in $\mathcal{Q}_A$. Hence we can assume without loss of generality that there is a relation $\alpha^t - \beta_1\beta_2\ldots\beta_r$ in I_A for some $t \in \mathbb{N}$ and arrows $\beta_1, \beta_2, \ldots, \beta_r$. Among all relations of this type we choose one with minimal t. Hence $(\alpha^t, \beta_1\beta_2\ldots\beta_r)$ is a contour in $\overrightarrow{A}$ with $t, r \geq 2$. Let $y = e(\beta_1)$ be the ending point of β_1 and $\tilde{\beta} = \beta_2 \ldots \beta_r$.

By the structure theorem for non-deep contours in [BGRS85, 6.4] the contour $(\alpha^t, \beta_1\beta_2\ldots\beta_r)$ is deep, i.e. we have $\alpha^{t+1} = 0$ in A. Since A is mild, the cardinality of the set x^+ of all arrows starting in x is bounded by three. Before we consider the cases $|x^+| = 2$ and $|x^+| = 3$ separately we shall prove some useful general facts.

The following trivial fact about standard algebras will be essential hereafter.

Lemma 3.1
Let $A = \overline{A}$ be a standard $\mathbf{k}$-algebra. Consider rays $v_i, w_j \in \overrightarrow{A} \setminus \{0\}$ for $i = 1 \ldots n$ and $j = 1 \ldots m$ such that $v_l \neq v_k$ and $w_l \neq w_k$ for $l \neq k$. If there are $\lambda_i, \mu_j \in \mathbf{k} \setminus \{0\}$ such that $\sum_{i=1}^n \lambda_i v_i = \sum_{j=1}^m \mu_j w_j$, then $n = m$ and there exists a permutation $\pi \in S(n)$ such that $v_i = w_{\pi(i)}$ and $\lambda_i = \mu_{\pi(i)}$ for $i = 1 \ldots n$.

Proof. Since the set of non-zero rays in $\overrightarrow{A}$ forms a basis of A, it is linearly independent and the claim follows. □

In what follows we denote by $\mathcal{L}$ the set of all long morphisms in $\overrightarrow{A}$. By μ we denote some long morphism $\nu\alpha^t\nu'$ which exists since $\alpha^t \neq 0$.

Lemma 3.2
Using the above notations we have:

$$\langle \beta_1 \rangle \cap \langle \alpha\beta_1 \rangle = 0$$

Proof. We assume to the contrary that $\langle \beta_1 \rangle \cap \langle \alpha\beta_1 \rangle \neq 0$. Then, by Lemma 3.1, there are rays $v, w \in \overrightarrow{A}$ such that $\beta_1 v = \alpha\beta_1 w \neq 0$. We claim that

$$D := \begin{tikzcd} \bullet \arrow[d, "\alpha^{t-1}"'] \arrow[dr, "\beta_1 w" near start] & \bullet \arrow[d, "v"] \arrow[dl, "\tilde{\beta}"' near start] \\ \bullet & \bullet \end{tikzcd}$$

is a cleaving diagram in $\overrightarrow{A}$. It is of representation-infinite, Euclidean type $\widetilde{A}_3$. Since all morphisms occurring in D are not long, the long morphism $\mu = \nu\alpha^t\nu'$ does not occur in D and D is still cleaving in $\overrightarrow{A}/\mu$ by [Bon09, Lemma 3]. Thus $\overrightarrow{A}/\mu$ is representation-infinite contradicting the mildness of A.

Now we show in detail, using [Bon09, Lemma 3 d)], that D is cleaving. First of all we assume that there is a ray ρ with $\rho\tilde{\beta} = \alpha^{t-1}$. Then we get $0 \neq \alpha^t = \alpha\rho\tilde{\beta} = \beta_1\tilde{\beta}$, whence $\alpha\rho = \beta_1$ by the cancellation law. This contradicts the fact that β_1 is an arrow. In a similar way it can be shown that $\rho\alpha^{t-1} = \tilde{\beta}$, $\rho v = \beta_1 w$ and $\rho\beta_1 w = v$ are impossible.
The following four cases are left to exclude.

i) $\alpha^{t-1}\rho = \beta_1 w$: Left multiplication with α gives us $\alpha^t\rho = \alpha\beta_1 w \neq 0$. Hence there is a non-deep contour $(\alpha^{t-1}\rho_1 \ldots \rho_k, \beta_1 w_1 \ldots w_l)$ in $\overrightarrow{A}$. Here $\rho = \rho_1 \ldots \rho_k$ resp. $w = w_1 \ldots w_l$ is a product of irreducible rays (arrows). Since the arrow β_1 is in the contour, the cycle $\beta_1\tilde{\beta}$ and the loop α belong to the contour. Hence it can only be a penny-farthing by the structure theorem for non-deep contours [BGRS85, 6.4]. But this case is excluded in the current section.

ii) $\tilde{\beta}\rho = v$: We argue as before and deduce $\beta_1\tilde{\beta}\rho = \beta_1 v = \alpha^t\rho = \alpha\beta_1 w \neq 0$. Hence there is a non-deep contour $(\alpha^{t-1}\rho_1 \ldots \rho_k, \beta_1 w_1 \ldots w_l)$ leading again to a contradiction.

iii) $\beta_1 w\rho = \alpha^{t-1}$: Since $t-1<t$ we have a contradiction to the minimality of t.

iv) $v\rho = \tilde{\beta}$: Then $\beta_1 v\rho = \beta_1\tilde{\beta} = \alpha^t = \alpha\beta_1 v\rho \neq 0$. Using the cancellation law we get $\alpha^{t-1} = \beta_1 v\rho$ a contradiction as before.

□

Lemma 3.3
*If $t \geq 3$ **and** $\mathcal{L} \nsubseteq \{\alpha^3, \alpha^2\beta_1\}$, then $\alpha^2\beta_1 = 0$.*

Proof. If $\alpha^2\beta_1 \neq 0$, then

$$D := \begin{tikzcd} \bullet \arrow[r, "\alpha"] & \bullet \arrow[r, "\beta_1"] \arrow[d, "\alpha"] & \bullet \\ \bullet & \bullet \arrow[l, "\alpha"'] \arrow[r, "\beta_1"] & \bullet \end{tikzcd}$$

is a cleaving diagram of Euclidian type $\widetilde{D}_5$ in $\overrightarrow{A}$. It is cleaving since:

i) $\alpha^2 = \beta_1\rho \neq 0$ contradicts the choice of $t \geq 3$.

ii) $\alpha\beta_1 = \beta_1\rho \neq 0$ contradicts Lemma 3.2.

It is also cleaving in $\overrightarrow{A}/\eta$ for $\eta \in \mathcal{L} \setminus \{\alpha^3, \alpha^2\beta_1\} \neq \emptyset$ contradicting the mildness of A. □

Lemma 3.4
If $\langle\alpha^2\rangle \cap \langle\alpha\beta_1\rangle = 0 = \langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle$, then $\langle\alpha^2, \beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0$.

Proof. Let $\alpha^2 u + \beta_1 v = \alpha\beta_1 w \neq 0$ be an element in $\langle\alpha^2, \beta_1\rangle \cap \langle\alpha\beta_1\rangle$. By Lemma 3.1 we can assume that u, v, w are rays and the following two cases might occur:

i) $\beta_1 v = \alpha\beta_1 w \neq 0$: This is a contradiction since $\langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0$.

ii) $\alpha^2 u = \alpha\beta_1 w \neq 0$: This is impossible because $\langle\alpha^2\rangle \cap \langle\alpha\beta_1\rangle = 0$.

□

3.2 The case $|x^+| = 2$

Lemma 3.5
If $x^+ = \{\alpha, \beta_1\}$ and $\mathcal{L} \subseteq \{\alpha^3, \alpha^2\beta_1\}$, then there exists an α-filtration $\mathcal{F}$ of P_x having finite projective dimension.

Proof. We treat two cases:

i) $\alpha\beta_1 = 0$: Then for $\langle\alpha^k\rangle$ with $k \geq 1$ only S_x is possible as a composition factor; hence $\mathrm{pdim}_\Lambda\langle\alpha^k\rangle < \infty$. Thus $P_x \supset \langle\alpha\rangle \supset \langle\alpha^2\rangle \supset \langle\alpha^3\rangle \supset 0$ is the wanted α-filtration.

ii) $\alpha\beta_1 \neq 0$: Since α^3 and $\alpha^2\beta_1$ are the only morphisms in $\overrightarrow{A}$ which can be long, we have $t=3$, $0 \neq \alpha^3 \in \mathcal{L}$, $\langle\alpha\beta_1\rangle = \mathbf{k}\,\alpha\beta_1 \cong S_y$ and $\langle\alpha^2\beta_1\rangle \in \{\mathbf{k}\,\alpha^2\beta_1,\ 0\}$.
Now we show that $\langle\alpha^2\rangle \cap \langle\alpha\beta_1\rangle = 0$. If there are rays $v = v_1 \ldots v_s,\ w \in \overrightarrow{A}$ with irreducible $v_i,\ i = 1\ldots, s$ such that $\alpha^2 v = \alpha\beta_1 w \neq 0$, then $s > 0$ because $s = 0$ would contradict the irreducibility of α. Therefore $v_1 = \alpha$ or $v_1 = \beta_1$.

- If $v_1 = \alpha$, then $v' = v_2 \ldots v_s = id$ since α^3 is long and $0 \neq \alpha^2 v = \alpha^3 v'$. Hence $0 \neq \alpha^3 = \alpha^2 v = \alpha\beta_1 w$ and $\alpha^2 = \beta_1 w$ contradicts the minimality of t.

- If $v_1 = \beta_1$, then $0 \neq \alpha^2 v = \alpha^2\beta_1 v' = \alpha\beta_1 w$; hence $0 \neq \alpha\beta_1 v' = \beta_1 w \in \langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0$.

Since $\langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0 = \langle\alpha^2\rangle \cap \langle\alpha\beta_1\rangle$, we deduce $\langle\beta_1, \alpha^2, \alpha\beta_1\rangle = \langle\beta_1, \alpha^2\rangle \oplus \langle\alpha\beta_1\rangle$ by Lemma 3.4. Therefore the graph of P_x has the following shape:

$$\begin{tikzcd}
e_x \arrow[d] \arrow[dr] & \\
\alpha \arrow[d] \arrow[dr] & \langle\beta_1\rangle \arrow[ddl, squiggly] \\
\alpha^2 \arrow[d] \arrow[dr] & \alpha\beta_1 \\
\alpha^3 & \alpha^2\beta_1
\end{tikzcd} .$$

Here $\langle\beta_1\rangle$ stands for the graph of the submodule $\langle\beta_1\rangle$ which is not known explicitly. Consider the module M defined by the following exact sequence:

$$0 \to \langle\beta_1, \alpha^2, \alpha\beta_1\rangle \to P_x \to M \to 0$$

Then $\operatorname{pdim}_\Lambda M < \infty$ since M is filtered by S_x and $\operatorname{pdim}_\Lambda(\langle\beta_1, \alpha^2\rangle \oplus \langle\alpha\beta_1\rangle) = \operatorname{pdim}_\Lambda\langle\beta_1, \alpha^2, \alpha\beta_1\rangle < \infty$. Thus $\operatorname{pdim}_\Lambda(\langle\alpha\beta_1\rangle \cong S_y)$ is finite too and the wanted α-filtration is $P_x \supset \langle\alpha\rangle \supset \langle\alpha^2\rangle \supset \langle\alpha^3\rangle \supset 0$.

□

Lemma 3.6
If $x^+ = \{\alpha, \beta_1\}$, $t \geq 3$ ***and*** *$\mathcal{L} \nsubseteq \{\alpha^3, \alpha^2\beta_1\}$, then $\alpha^2\rho = 0$ for all rays $\rho \notin \{e_x, \alpha, \ldots, \alpha^{t-2}\}$. Moreover, $\langle\alpha^2\rangle \cap \langle\alpha\beta_1\rangle = 0$.*

Proof. Let $\rho \in \overrightarrow{A}$ with $\alpha^2\rho \neq 0$ be written as a composition of irreducible rays $\rho = \rho_1 \ldots \rho_s$. Then the following two cases are possible:

i) $\rho = \alpha^s$: Since $0 \neq \alpha^2\rho = \alpha^{2+s}$ and $\alpha^{t+1} = 0$ we have $s \leq t-2$ and $\rho = \alpha^s \in \{e_x, \alpha, \ldots, \alpha^{t-2}\}$.

ii) There exists a minimal $1 \leq i \leq s$ such that $\rho_i \neq \alpha$. Since $x^+ = \{\alpha, \beta_1\}$, we have $\rho_i = \beta_1$ and $0 \neq \alpha^2\rho = \alpha^{2+i-1}\beta_1\rho_{i+1}\ldots\rho_s = 0$ by Lemma 3.3.

If $0 \neq \alpha^2 v = \alpha\beta_1 w$, then $v = \alpha^s$ with $0 \leq s \leq t-2$. Hence $0 = \alpha^2 v = \alpha^{s+2} = \alpha\beta_1 w$ and $\alpha^{s+1} = \beta_1 w$ by cancellation law. This contradicts the minimality of t. □

Corollary 3.7
If $x^+ = \{\alpha, \beta_1\}$, $t \geq 3$ ***and*** *$\mathcal{L} \nsubseteq \{\alpha^3, \alpha^2\beta_1\}$, then $\langle\alpha^2, \beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0$.*

Proof. The claim is trivial using Lemmas 3.2, 3.4 and 3.6. □

Proposition 3.8
If $x^+ = \{\alpha, \beta_1\}$, then there exists an α-filtration $\mathcal{F}$ of P_x having finite projective dimension.

Proof. If $\mathcal{L} \subseteq \{\alpha^3, \alpha^2\beta_1\}$, then the claim is the statement of Lemma 3.5. If $\mathcal{L} \nsubseteq \{\alpha^3, \alpha^2\beta_1\}$, then we consider the value of t:

i) $t = 2$: Then the graph of P_x has the following shape:

$$\begin{tikzcd}
e_x \arrow[d] \arrow[dr] & \\
\alpha \arrow[d] \arrow[dr] & \langle \beta_1 \rangle \arrow[dl] \\
\alpha^2 \arrow[dr] & \langle \alpha\beta_1 \rangle \\
 & \langle \alpha^2\beta_1 \rangle
\end{tikzcd} .$$

Let a subquotient M of P_x be defined by the following exact sequence:

$$0 \to \langle \beta_1, \alpha\beta_1 \rangle \to P_x \to M \to 0$$

Then M and $\langle \beta_1, \alpha\beta_1 \rangle$ have finite projective dimension in mod-Λ. By Lemma 3.2 we have $\langle \beta_1, \alpha\beta_1 \rangle = \langle \beta_1 \rangle \oplus \langle \alpha\beta_1 \rangle$; hence $\mathrm{pdim}_\Lambda \langle \beta_1 \rangle$ and $\mathrm{pdim}_\Lambda \langle \alpha\beta_1 \rangle$ are both finite.
Let K be the kernel of the epimorphism $\lambda_\alpha : \langle \beta_1 \rangle \to \langle \alpha\beta_1 \rangle$, $\lambda_\alpha(\rho) = \alpha\rho$. Then $\mathrm{pdim}_\Lambda K < \infty$ and for the α-filtration $\mathcal{F}$ we take the following: $P_x \supset \langle \alpha, \beta_1 \rangle \supset \langle \beta_1 \rangle \oplus \langle \alpha\beta_1 \rangle \supset \langle \alpha\beta_1 \rangle \oplus K \supset K \supset 0$.

ii) $t \geq 3$: Consider the following exact sequences:

$$0 \to \langle \alpha, \beta_1 \rangle \to P_x \to S_x \to 0$$

$$0 \to \langle \alpha^2, \beta_1, \alpha\beta_1 \rangle \to \langle \alpha, \beta_1 \rangle \to S_x \to 0$$

Hence $\mathrm{pdim}_\Lambda \langle \alpha, \beta_1 \rangle$ and $\mathrm{pdim}_\Lambda \langle \alpha^2, \beta_1, \alpha\beta_1 \rangle$ are finite. By Corollary 3.7 $\langle \alpha^2, \beta_1, \alpha\beta_1 \rangle = \langle \alpha^2, \beta_1 \rangle \oplus \langle \alpha\beta_1 \rangle$, that means $\mathrm{pdim}_\Lambda \langle \alpha\beta_1 \rangle$ is finite too. With Lemma 3.6 it is easily seen that for $2 \leq k \leq t$ the module $\langle \alpha^k \rangle$ is a uniserial module with S_x as the only composition factor. Hence $\mathrm{pdim}_\Lambda \langle \alpha^k \rangle$ is finite for $2 \leq k \leq t$. Thereby we have the wanted α-filtration

$$P_x \supset \langle \alpha, \beta_1 \rangle \supset \langle \alpha^2 \rangle \oplus \langle \alpha\beta_1 \rangle \supset \langle \alpha^3 \rangle \supset \langle \alpha^4 \rangle \supset \ldots \supset \langle \alpha^t \rangle \supset 0.$$

□

3.3 The case $|x^+| = 3$

With previous notations $x^+ = \{\alpha, \beta_1, \gamma\}$, $(\alpha^t, \beta_1\beta_2 \ldots \beta_r)$ is a contour in $\overrightarrow{A}$, $t \geq 2$, $\alpha^{t+1} = 0$, $\tilde{\beta} := \beta_2 \ldots \beta_r$ and $\mu = \nu\alpha^t\nu'$ is a long morphism in $\overrightarrow{A}$.

The α-filtrations will be constructed depending on the set $\mathcal{L}$ of long morphisms in $\overrightarrow{A}$. The case $\mathcal{L} \subseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$ is treated in Lemma 3.16, the case $\mathcal{L} \subseteq \{\alpha^t, \alpha^2\beta_1\}$ in 3.17 and the remaining case in 3.18.

But first, we derive some technical results.

Lemma 3.9
If $r = 2$ and $\delta : z' \to z$ is an arrow in $\mathcal{Q}_A$ ending in $z = e(\gamma)$, then $\delta = \gamma$.

Proof. Assume to the contrary that $\gamma \neq \delta : z' \to z$, then there is no arrow $\beta_1 \neq \varepsilon : y' \to y$ in $\mathcal{Q}_\Lambda$. If there is such an arrow, then by the definition of a neighborhood ε belongs to $\mathcal{Q}_A$. This arrow induces

an irreducible ray $\beta_1 \neq \varepsilon : y' \to y$ in $\overrightarrow{A}$ and

$$D := \begin{tikzcd} \bullet \arrow[r, "\delta"] & \bullet & \bullet \arrow[l, "\gamma"'] \arrow[r, "\beta_1"] \arrow[d, "\alpha"] & \bullet & \bullet \arrow[l, "\varepsilon"'] \\ & & \bullet & & \\ & & \bullet \arrow[u, "\beta_2"'] & & \end{tikzcd}$$

is a cleaving diagram in $\overrightarrow{A}/\mu$ of Euclidian type $\widetilde{E}_6$.
In a similar way an arrow $\alpha, \beta_2 \neq \varepsilon : x' \to x$ in $\mathcal{Q}_\Lambda$ leads to a cleaving diagram of type $\widetilde{D}_5$ in $\overrightarrow{A}/\mu$. Hence the full subcategory B of Λ supported by the points $x,\ y$ is a convex subcategory of Λ. Therefore the projective dimensions of S_x is finite in mod-B since it is finite in mod-Λ. But in B we have $x^+ = \{\alpha, \beta_1\}$, whence we can apply Proposition 3.8 together with 1.3 to get the contradiction that α is not a loop. $\square$

Lemma 3.10
If $\alpha\gamma \neq 0$, then $\beta_1 v \neq \alpha\gamma \neq \gamma w$ for all rays $v, w \in \overrightarrow{A}$.

Proof. i) Assume that there exists a ray $v \in \overrightarrow{A}$ such that $\beta_1 v = \alpha\gamma \neq 0$. Then

$$D := \begin{tikzcd} \bullet \arrow[d, "\gamma"'] \arrow[dr, "\alpha^{t-1}" near start] & \bullet \arrow[d, "\tilde{\beta}"] \arrow[dl, "v"' near start] \\ \bullet & \bullet \end{tikzcd}$$

is a cleaving diagram of Euclidian type $\widetilde{A}_3$ in $\overrightarrow{A}/\mu$.

- For $\gamma\rho = \alpha^{t-1}$ or $v\rho = \tilde{\beta}$ we have $\alpha\gamma\rho = \beta_1 v\rho = \beta_1\tilde{\beta} = \alpha^t \neq 0$. Thus $\alpha^{t-1} = \gamma\rho$ contradicts the choice of t.
- If $\alpha^{t-1}\rho = \gamma$ or $\tilde{\beta}\rho = v$, then $\alpha^t\rho = \beta_1\tilde{\beta}\rho = \beta_1 v = \alpha\gamma \neq 0$. Then $\alpha^{t-1}\rho = \gamma$ contradicts the irreducibility of γ.

ii) Assume that there exists a ray $w = w_1 \dots w_s : z \rightsquigarrow z \in \overrightarrow{A}$ with irreducible w_i such that $\gamma w = \alpha\gamma \neq 0$.

$r = 2$: Since w_s is an irreducible ray ending in z, $w_s = \gamma$ by Lemma 3.9. Thus we get a contradiction $\gamma w_1 \dots w_{s-1} = \alpha$.

$r \geq 3$: We look at the value of s. If $s = 1$, then $w = w_1$ is a loop and

$$D := \begin{tikzcd} \bullet \arrow[r, "w_1 = w"] & \bullet \arrow[ddr, dotted, bend right, "(1)"'] & \bullet \arrow[l, "\gamma"'] \arrow[r, "\alpha"] \arrow[d, "\beta_1"] & \bullet \arrow[ddl, dotted, bend left, "(2)"] & \bullet \arrow[l, "\beta_r"'] \\ & & \bullet \arrow[d, "\beta_2"] & & \\ & & \bullet & & \end{tikzcd}$$

is a cleaving diagram in $\overrightarrow{A}/\mu$.
If $s \geq 2$, then

$$D := \begin{tikzcd} \bullet \arrow[r, "w_{s-1}"] \arrow[rrr, dotted, bend right, "(3)"'] & \bullet \arrow[r, "w_s"] & \bullet & \bullet \arrow[l, "\gamma"'] \arrow[r, "\alpha"] \arrow[d, "\beta_1"] & \bullet & \bullet \arrow[l, "\beta_r"'] & \bullet \arrow[l, "\beta_{r-1}"'] \arrow[lll, dotted, bend left, "(4)"] \\ &&& \bullet &&& \end{tikzcd}$$

is cleaving in $\overrightarrow{A}/\mu$.
We still have to show that not any morphisms indicated by the dotted lines make the diagrams commute.

(1): $\gamma\rho = \beta_1\beta_2$, with $\rho = \rho_1 \ldots \rho_l$. If $\rho = w_1^l = w^l$, then $\beta_1\beta_2 = \gamma\rho = \gamma w^l = \alpha\gamma w^{l-1}$ and $\beta_1\beta_2 \ldots \beta_r = \alpha^t = \alpha\gamma w^{l-1}\beta_3 \ldots \beta_r \neq 0$. Therefore $\alpha^{t-1} = \gamma w^{l-1}\beta_3 \ldots \beta_r$ is a contradiction. If $\rho \neq w_1^l$, then one of the irreducible rays $\rho_i \neq w_1$ starts in z and

$$D := \begin{tikzcd} \bullet & \bullet \arrow[l, "\rho_i"'] \arrow[r, "w_1"] & \bullet & \bullet \arrow[l, "\gamma"'] \arrow[r, "\alpha"] \arrow[d, "\beta_1"] & \bullet & \bullet \arrow[l, "\beta_r"'] & \bullet \arrow[l, "\beta_{r-1}"'] \arrow[lll, dotted, bend left, "(4)"] \\ &&& \bullet &&& \end{tikzcd}$$

is cleaving in $\overrightarrow{A}/\mu$.

(2): If $\alpha\rho = \beta_1\beta_2$, then $\alpha\rho\beta_3 \ldots \beta_r = \beta_1\beta_2 \ldots \beta_r = \alpha^t \neq 0$ and $\alpha^{t-1} = \rho\beta_3 \ldots \beta_r$ contradicts the minimality of t.

(3): If $\rho\gamma = w_{s-1}w_s$, then $\gamma w_1 \ldots w_{s-2}\rho\gamma = \gamma w = \alpha\gamma \neq 0$ and $\alpha = \gamma w_1 \ldots w_{s-2}\rho$ contradicts the irreducibility of α.

(4): If $\rho\alpha = \beta_{r-1}\beta_r$, then $\beta_1\beta_2 \ldots \beta_{r-2}\rho\alpha = \beta_1\beta_2 \ldots \beta_r = \alpha^t \neq 0$ and $\alpha^{t-1} = \beta_1\beta_2 \ldots \beta_{r-2}\rho$ contradicts the minimality of t.

□

Lemma 3.11
If $t \geq 3$, then $\alpha\gamma = 0$.

Proof. Assume that $\alpha\gamma \neq 0$, then

$$D := \begin{tikzcd} \bullet & \bullet \arrow[l, "\gamma"'] \arrow[r, "\beta_1"] \arrow[d, "\alpha"] & \bullet \\ \bullet & \bullet \arrow[l, "\gamma"'] \arrow[r, "\alpha"] & \bullet \end{tikzcd}$$

is a cleaving diagram of Euclidian type in $\overrightarrow{A}/\mu$. It is cleaving since:

i) $\gamma\rho = \alpha\gamma$ or $\beta_1\rho = \alpha\gamma$ contradicts Lemma 3.10,

ii) $\gamma\rho = \alpha^2$ or $\beta_1\rho = \alpha^2$ contradicts the minimality of $t \geq 3$.

□

Lemma 3.12

a) If $\mathcal{L} \nsubseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$, then $\alpha\beta_1 = 0$ or $\alpha\gamma = 0$.

b) If $\alpha^2\beta_1 \neq 0$, then $\gamma w \neq \alpha\beta_1$ for all $w \in \overrightarrow{A}$.

Proof. a) If $\alpha\beta_1 \neq 0$ and $\alpha\gamma \neq 0$, then

$$D := \begin{tikzcd} & \bullet \arrow[d, "\alpha"] & \\ \bullet & \bullet \arrow[l, "\gamma"'] \arrow[r, "\beta_1"] \arrow[d, "\alpha"] & \bullet \\ & \bullet & \end{tikzcd}$$

is a cleaving diagram of Euclidian type $\widetilde{D}_4$ in $\overrightarrow{A}$. It is still cleaving in $\overrightarrow{A}/\eta$ for $\eta \in \mathcal{L}\backslash\{\alpha^2, \alpha\beta_1, \alpha\gamma\} \neq \emptyset$.

b) Since $\alpha^2\beta_1 \neq 0$, we have $\alpha\gamma = 0$ by *a*). But $\gamma w = \alpha\beta_1$ leads to the contradiction $0 \neq \alpha^2\beta_1 = \alpha\gamma w = 0$.

$\square$

Lemma 3.13
If $t = 2$ ***or*** $\mathcal{L} \nsubseteq \{\alpha^t, \alpha^2\beta_1\}$, *then:*

a) $\alpha^2\beta_1 = 0 = \alpha^2\gamma$, $\alpha^2\rho = 0$ *for all rays* $\rho \notin \{e_x, \alpha, \dots, \alpha^{t-2}\}$.

b) $\langle\beta_1\rangle \cap \langle\alpha\gamma\rangle = 0$.

c) If $\langle\gamma\rangle \cap \langle\beta_1\rangle = 0$, *then* $\langle\gamma\rangle \cap \langle\alpha^2\rangle = 0$.

d) $\langle\gamma\rangle \cap \langle\alpha^t\rangle = 0$ *or* $\langle\gamma\rangle \cap \langle\alpha\beta_1\rangle = 0$.

e) $\langle\gamma\rangle \cap \langle\alpha\beta_1\rangle = 0$ *or* $\langle\gamma\rangle \cap \langle\beta_1\rangle = 0$.

f) $\langle\alpha\beta_1\rangle \cap \langle\alpha^2\rangle = 0$ *and* $\langle\alpha\gamma\rangle \cap \langle\alpha^2\rangle = 0$.

Proof. a) Consider the case $t = 2$.

i) If $\alpha^2\beta_1 \neq 0$, then $\beta_r\beta_1 \neq 0$ and

$$\begin{tikzcd} \bullet & \bullet \arrow[l, "\gamma"'] \arrow[r, "\beta_1"] \arrow[d, "\alpha"] & \bullet \\ \bullet & \bullet \arrow[l, "\beta_1"'] & \bullet \arrow[l, "\beta_r"'] \end{tikzcd}$$

is a cleaving diagram of Euclidian type $\widetilde{D}_5$ in $\overrightarrow{A}/\mu$. The diagram is cleaving because:

- $\beta_1\rho = \alpha\beta_1 \neq 0$ is a contradiction of Lemma 3.2,
- $\gamma\rho = \alpha\beta_1 \neq 0$ contradicts Lemma 3.12 b).

ii) If $\alpha^2\gamma \neq 0$, then $\beta_r\gamma \neq 0$ and

$$\begin{tikzcd} \bullet & \bullet \arrow[l, "\gamma"'] \arrow[r, "\beta_1"] \arrow[d, "\alpha"] & \bullet \\ \bullet & \bullet \arrow[l, "\gamma"'] & \bullet \arrow[l, "\beta_r"'] \end{tikzcd}$$

is a cleaving diagram in $\overrightarrow{A}/\mu$. It is cleaving since $\beta_1\rho = \alpha\gamma$ resp. $\gamma\rho = \alpha\gamma$ contradicts Lemma 3.10.

In the case $t \geq 3$, $\alpha^2\gamma = 0$ by Lemma 3.11. If $t = 3$, then $\mathcal{L} \nsubseteq \{\alpha^3, \alpha^2\beta_1\}$ by assumption. If $t > 3$, then $\mu = \nu\alpha^t\nu' \in \mathcal{L} \setminus \{\alpha^3, \alpha^2\beta_1\}$. Hence $\alpha^2\beta_1 = 0$ by Lemma 3.3 in both cases.

b) If v, w are rays in $\overrightarrow{A}$ such that $\beta_1 v = \alpha\gamma w \neq 0$, then the diagram

$$D := \begin{tikzcd} \bullet \arrow[d, "\gamma w"'] \arrow[dr, "\alpha^{t-1}" near start] & \bullet \arrow[d, "\tilde{\beta}"] \arrow[dl, "v"' near start] \\ \bullet & \bullet \end{tikzcd}$$

is a cleaving diagram in $\overrightarrow{A}/\mu$.

i) If $\gamma w\rho = \alpha^{t-1}$ or $v\rho = \tilde{\beta}$, then $\beta_1 v\rho = \beta_1\tilde{\beta} = \alpha^t = \alpha\gamma w\rho \neq 0$. Hence $\gamma w\rho = \alpha^{t-1}$ contradicts the minimality of t.

ii) If $\alpha^{t-1}\rho = \gamma w$ or $\tilde{\beta}\rho = v$, then $0 \neq \beta_1 v = \beta_1\tilde{\beta}\rho = \alpha\gamma w = \alpha^t\rho = 0$ by a).

c) Let v, w be rays such that $\gamma v = \alpha^2 w \neq 0$. By *a*) we have $w = \alpha^k$ with $0 \leq k \leq t-2$, that means $\gamma v = \alpha^{2+k}$. Since t is minimal, we have $t = 2+k$ and $0 \neq \gamma v = \alpha^t = \beta_1\tilde{\beta} \in \langle\gamma\rangle \cap \langle\beta_1\rangle = 0$.

d) Let v, w, v', w' be rays in $\overrightarrow{A}$ such that $\gamma w = \alpha^t v \neq 0$ and $\gamma w' = \alpha\beta_1 v' \neq 0$. Then

$$D := \begin{tikzcd} \bullet \arrow[d, "w"'] \arrow[dr, "w'" near start] & \bullet \arrow[d, "\beta_1 v'"] \arrow[dl, "\alpha^{t-1}v"' near start] \\ \bullet & \bullet \end{tikzcd}$$

is a cleaving diagram in $\overrightarrow{A}/\mu$.

i) If $w\rho = w'$ or $\alpha^{t-1}v\rho = \beta_1 v'$, then $\gamma w\rho = \gamma w' = \alpha^t v\rho = \alpha\beta_1 v' \neq 0$. Hence there is a non-deep contour $(\alpha^{t-1}v_1 \ldots v_k\rho_1 \ldots \rho_l, \beta_1 v_1' \ldots v_s')$ in $\overrightarrow{A}$ which can only be a penny-farthing by the structure theorem for non-deep contours. But this case is excluded in the current section.

ii) If $w'\rho = w$ or $\beta_1 v'\rho = \alpha^{t-1}v$, then $\gamma w'\rho = \gamma w = \alpha\beta_1 v'\rho = \alpha^t v \neq 0$. Again, we have a non-deep contour $(\alpha^{t-1}v_1 \ldots v_k, \beta_1 v_1' \ldots v_l'\rho_1 \ldots \rho_s)$ which leads to a contradiction as before.

e) Let v, w, v', w' be rays such that $\beta_1 v = \gamma w \neq 0$ and $\alpha\beta_1 v' = \gamma w' \neq 0$. Then

$$\begin{tikzcd} \bullet \arrow[d, "w"'] \arrow[dr, "w'" near start] & \bullet \arrow[dl, "v"' near start] \arrow[dr, "\tilde{\beta}" near start] & \bullet \arrow[d, "\alpha^{t-1}"] \arrow[dl, "\beta_1 v'"' near start] \\ \bullet & \bullet & \bullet \end{tikzcd}$$

is a cleaving diagram in $\overrightarrow{A}/\mu$.

i) If $w\rho = w'$, we get the contradiction $0 \neq \gamma w\rho = \gamma w' = \beta_1 v\rho = \alpha\beta_1 v' \in \langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0$.

ii) If $w'\rho = w$, then $0 \neq \gamma w'\rho = \gamma w = \alpha\beta_1 v'\rho = \beta_1 v \in \langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0$.

iii) If $v\rho = \tilde{\beta}$, then $0 \neq \beta_1 v\rho = \beta_1\tilde{\beta} = \gamma w\rho = \alpha^t \in \langle\gamma\rangle \cap \langle\alpha^t\rangle = 0$ by d).

iv) If $\tilde{\beta}\rho = v$, then $0 \neq \beta_1\tilde{\beta}\rho = \beta_1 v = \alpha^t\rho = \gamma w \in \langle\gamma\rangle \cap \langle\alpha^t\rangle = 0$ by d).

v) If $\alpha^{t-1}\rho = \beta_1 v'$, then $0 \neq \alpha^t\rho = \alpha\beta_1 v' = \gamma w' \in \langle\gamma\rangle \cap \langle\alpha^t\rangle = 0$ by d).

vi) The case $\beta_1 v'\rho = \alpha^{t-1}$ contradicts the minimality of t.

f) If v, w are rays in $\overrightarrow{A}$ such that $\alpha\beta_1 v = \alpha^2 w \neq 0$ resp. $\alpha\gamma v = \alpha^2 w \neq 0$, then $w = \alpha^k$ with $0 \leq k \leq t-2$ and $\beta_1 v = \alpha^{1+k}$ resp. $\gamma v = \alpha^{1+k}$. Since t is minimal, we get the contradiction $t = 1+k < t$.

$\square$

Lemma 3.14
If $\mathcal{L} \nsubseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$, then $\langle\gamma\rangle \cap \langle\alpha\gamma\rangle = 0$.

Proof. In the case $t \geq 3$, the claim is trivial since $\alpha\gamma = 0$ by 3.11.

Consider the case $t = 2$. Assume that there exist rays v, w in $\overrightarrow{A}$ such that $\gamma v = \alpha\gamma w \neq 0$. First of all, we deduce that $w \neq id$ by Lemma 3.10 and $v \neq id$ since γ is an arrow. Therefore we can write $v = v_1 \dots v_s$, $w = w_1 \dots, w_q$ with irreducible rays $v_i, w_j \in \overrightarrow{A}$. Consider the value of q:

a) If $q = 1$, then the diagram

$$\begin{tikzcd}
\bullet \arrow[r, "v_s"] & \bullet & \bullet \arrow[l, "w_1=w"'] & \bullet \arrow[l, "\gamma"'] \arrow[r, "\beta_1"] \arrow[d, "\alpha"'] & \bullet \\
& & & \bullet \arrow[d, "\gamma"'] & \bullet \arrow[l, "\beta_r"'] \\
& & & \bullet &
\end{tikzcd}$$

is a cleaving diagram of Euclidian type $\widetilde{E}_7$ in $\overrightarrow{A}/\mu$ (see [GR92, 10.7]).

b) If $q \geq 2$, then the diagram

$$\begin{tikzcd}
\bullet & \bullet \arrow[l, "w_2 \dots w_q"'] & \bullet \arrow[l, "w_1"'] & \bullet \arrow[l, "\gamma"'] \arrow[r, "\beta_1"] \arrow[d, "\alpha"'] & \bullet \\
& & & \bullet \arrow[d, "\gamma"'] & \bullet \arrow[l, "\beta_r"'] \\
& & & \bullet &
\end{tikzcd}$$

is cleaving in $\overrightarrow{A}/\mu$.

The diagrams are cleaving because:

i) $\alpha\rho = \gamma w \neq 0$: Then $0 \neq \alpha\gamma w = \alpha^2\rho = 0$ by Lemma 3.13 a).

ii) $\gamma\rho = \alpha\gamma \neq 0$ contradicts Lemma 3.10.

iii) $\beta_1\rho = \gamma w \neq 0$: Then $0 \neq \alpha\gamma w = \alpha\beta_1\rho = 0$ since $\alpha\beta_1 = 0$ by Lemma 3.12.

iv) $\rho v_s = \gamma w \neq 0$: Then $\alpha\rho v_s = \alpha\gamma w \neq 0$. If $\rho = \beta_1\rho'$, then $0 = \alpha\beta_1\rho' v_s = \alpha\gamma w \neq 0$. If $\rho = \gamma\rho'$, then $\alpha\gamma\rho' v_s = \alpha\gamma w$ and $w_1 = w = \rho' v_s$. Hence $\rho' = id$ and $v_s = w_1$. Therefore $0 \neq \gamma v = \gamma v_1 \dots v_{s-1} w_1 = \alpha\gamma w_1$ and $\gamma v_1 \dots v_{s-1} = \alpha\gamma$ contradicting Lemma 3.10. If $\rho = \alpha\rho'$, then $0 \neq \alpha\gamma w = \alpha^2\rho' v_s = 0$ by Lemma 3.13 a).

v) $\beta_1\rho = \alpha\gamma \neq 0$ contradicts Lemma 3.10.

□

Lemma 3.15
Let $\mathcal{L} \nsubseteq \{\alpha^t, \alpha^2\beta_1\}$ and $\mathcal{L} \nsubseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$.

a) If $\langle\alpha\gamma\rangle = 0 = \langle\gamma\rangle \cap \langle\alpha\beta_1\rangle$, then $\langle\beta_1, \gamma, \alpha^2\rangle \cap \langle\alpha\beta_1\rangle = 0$.

b) If $\langle\alpha\gamma\rangle = 0 = \langle\gamma\rangle \cap \langle\beta_1\rangle$, then $\langle\beta_1, \alpha^2\rangle \cap \langle\gamma, \alpha\beta_1\rangle = 0$.

c) If $\langle\alpha\beta_1\rangle = 0$, then $\langle\beta_1, \gamma, \alpha^2\rangle \cap \langle\alpha\gamma\rangle = 0$.

Proof. We only prove b); the other cases are proven analogously. Let $v, v', w, w' \in A$ be such that $\beta_1 v + \alpha^2 v' = \gamma w + \alpha\beta_1 w' \neq 0$. That means we have rays $v_i, w_j \in \overrightarrow{A}$, numbers $\lambda_i, \mu_j \in \mathbf{k}$ and integers $s_1, s_2 \geq 0,\ n_1, n_2 \geq 1$ such that

$$\sum_{i=1}^{s_1} \lambda_i\, \beta_1 v_i + \sum_{i=s_1+1}^{n_1} \lambda_i\, \alpha^2 v_i = \sum_{j=1}^{s_2} \mu_j \gamma w_j + \sum_{j=s_2+1}^{n_2} \mu_j \alpha\beta_1 w_j$$

and $\beta_1 v_i \neq \beta_1 v_j,\ \alpha^2 v_i \neq \alpha^2 v_j,\ \gamma w_i \neq \gamma w_j,\ \alpha\beta_1 w_i \neq \alpha\beta_1 w_j$ for $i \neq j$. Without loss of generality we can assume that all λ_i, μ_j are non-zero, that $\beta_1 v_i \neq \alpha^2 v_j$ for $i = 1 \ldots s_1,\ j = s_1 + 1 \ldots n_1$ and $\gamma w_i \neq \alpha\beta_1 w_j$ for $i = 1 \ldots s_2,\ j = s_2 + 1 \ldots n_2$. Then by Lemma 3.1 we have $n_1 = n_2$ and there exists a permutation π such that $\beta_1 v_i = \gamma w_{\pi(i)} \in \langle \beta_1 \rangle \cap \langle \gamma \rangle = 0$ or $\beta_1 v_i = \alpha\beta_1 w_{\pi(i)} \in \langle \beta_1 \rangle \cap \langle \alpha\beta_1 \rangle = 0$ by Lemma 3.2. Hence $s_1 = 0$. Moreover, by Lemma 3.13 we have $\alpha^2 v_i = \gamma w_{\pi(i)} \in \langle \alpha^2 \rangle \cap \langle \gamma \rangle = 0$ or $\alpha^2 v_i = \alpha\beta_1 w_{\pi(i)} \in \langle \alpha^2 \rangle \cap \langle \alpha\beta_1 \rangle = 0$; this is possible for $n_1 - s_1 = 0$ only. Hence $n_1 = 0$, contradicting the choice of n_1. $\square$

Lemma 3.16
If $\mathcal{L} \subseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$, then there exists an α-filtration $\mathcal{F}$ of P_x having finite projective dimension.

Proof. Since $\mathcal{L} \subseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$, $\mu = \alpha^2$ is long and $t = 2$. Now it is easily seen that $\langle \alpha^2 \rangle = \mathbf{k}\,\alpha^2 \cong S_x$, $\langle \alpha\gamma \rangle = \mathbf{k}\,\alpha\gamma$, $\langle \alpha\beta_1 \rangle = \mathbf{k}\,\alpha\beta_1$ and $\langle \alpha \rangle$ has a $\mathbf{k}$ basis $\{\alpha, \alpha^2, \alpha\beta_1, \alpha\gamma\}$. Using Lemma 3.2 and 3.10 we conclude $\langle \beta_1 \rangle \cap \langle \alpha\beta_1 \rangle = 0$ and $\langle \gamma \rangle \cap \langle \alpha\gamma \rangle = 0 = \langle \beta_1 \rangle \cap \langle \alpha\gamma \rangle$.
By Lemma 3.13 d) $\langle \gamma \rangle \cap \langle \alpha^2 \rangle = 0$ or $\langle \gamma \rangle \cap \langle \alpha\beta_1 \rangle = 0$. Thus the graph of P_x has one of the following shapes:

$$\begin{tikzcd}
 & e_x \arrow[dl] \arrow[d] \arrow[dr] & \\
\langle\gamma\rangle \arrow[dr, bend right] & \alpha \arrow[dl] \arrow[d] \arrow[dr] & \langle\beta_1\rangle \arrow[dl, bend left] \\
\alpha\gamma & \alpha^2 & \alpha\beta_1
\end{tikzcd}
\quad or \quad
\begin{tikzcd}
 & e_x \arrow[dl] \arrow[d] \arrow[dr] & \\
\langle\gamma\rangle \arrow[drr, bend left] & \alpha \arrow[dl] \arrow[d] \arrow[dr] & \langle\beta_1\rangle \arrow[dl, bend left] \\
\alpha\gamma & \alpha^2 & \alpha\beta_1
\end{tikzcd}
.$$

In the first case we consider the following exact sequence:

$$0 \to \langle \alpha^2 \rangle \to \langle \alpha, \beta_1, \gamma \rangle \to \langle \alpha, \beta_1, \gamma \rangle / \langle \alpha^2 \rangle \to 0$$

Since $\langle \alpha \rangle$ has $\mathbf{k}$ basis $\{\alpha, \alpha^2, \alpha\beta_1, \alpha\gamma\rangle$ and $\mathcal{L} \subseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$ we have $\langle \alpha, \beta_1, \gamma \rangle / \langle \alpha^2 \rangle = \langle \alpha \rangle / \langle \alpha^2 \rangle \oplus \langle \beta_1, \gamma \rangle / \langle \alpha^2 \rangle$. Hence $\operatorname{pdim}_\Lambda \langle \alpha \rangle < \infty$ and $P_x \supset \langle \alpha \rangle \supset \langle \alpha^2 \rangle \supset 0$ is the wanted filtration.

In the second case we have $\langle \alpha, \beta_1, \gamma \rangle / \langle \alpha^2 \rangle = \langle \alpha, \gamma \rangle / \langle \alpha^2 \rangle \oplus \langle \beta_1 \rangle / \langle \alpha^2 \rangle$. Thus $\operatorname{pdim}_\Lambda \langle \alpha, \gamma \rangle < \infty$. Now we consider

$$0 \to \langle \beta_1, \gamma, \alpha\gamma \rangle \to \langle \alpha, \beta_1, \gamma \rangle \to S_x \to 0.$$

Since $\langle \beta_1, \gamma, \alpha\gamma \rangle = \langle \beta_1, \gamma \rangle \oplus \langle \alpha\gamma \rangle$, we have $\operatorname{pdim}_\Lambda \langle \alpha\gamma \rangle < \infty$ and $P_x \supset \langle \alpha, \gamma \rangle \supset \langle \alpha^2, \alpha\gamma \rangle \supset 0$ is a suitable filtration. $\square$

Lemma 3.17
If $\mathcal{L} \subseteq \{\alpha^t, \alpha^2\beta_1\}$, then there exists an α-filtration $\mathcal{F}$ of P_x having finite projective dimension.

Proof. If $t = 2$, then $\alpha^2\beta_1 = 0$ by Lemma 3.13 a). Hence $\mathcal{L} \subseteq \{\alpha^2\}$ and the filtration exists by Lemma 3.16.
If $t \geq 3$, then $\alpha\gamma = 0$ by Lemma 3.11. From the assumption $\mathcal{L} \subseteq \{\alpha^t, \alpha^2\beta_1\}$ it is easily seen that $\langle \alpha\beta_1 \rangle = \mathbf{k}\,\alpha\beta_1$ and $\langle \alpha^2\beta_1 \rangle = \mathbf{k}\,\alpha^2\beta_1$.

i) If $\alpha^2\beta_1 = 0$, then α^t is the only long morphism in $\overrightarrow{A}$; hence $\alpha\beta_1 = 0$ and $\langle \alpha^k \rangle$, $k \geq 1$, is uniserial of finite projective dimension. Thus $P_x \supset \langle \alpha \rangle \supset \langle \alpha^2 \rangle \supset \ldots \supset \langle \alpha^t \rangle \supset 0$ is a suitable α-filtration.

ii) If $\alpha^2\beta_1 \neq 0$, then $\langle\alpha\beta_1\rangle = \mathbf{k}\,\alpha\beta_1 \cong S_y \cong \langle\alpha^2\beta_1\rangle$. By 3.2 and 3.12 b) $\langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0 = \langle\gamma\rangle \cap \langle\alpha\beta_1\rangle$. Therefore the graph of P_x has the following shape:

$$
\begin{tikzcd}
 & e_x \arrow[dl] \arrow[d] \arrow[dr] & \\
\langle\gamma\rangle \arrow[ddr, bend right=20] \arrow[dddr, bend right=60] & \alpha \arrow[d] \arrow[dr] & \langle\beta_1\rangle \arrow[dddl, bend left=60] \\
 & \alpha^2 \arrow[d] \arrow[dr] & \alpha\beta_1 \\
 & \alpha^3 \arrow[d, dotted] & \alpha^2\beta_1 \\
 & \alpha^t &
\end{tikzcd}
$$

Moreover, $\langle\alpha\beta_1\rangle \cong S_y$ is a direct summand of the module $\langle\alpha^2, \beta_1, \gamma, \alpha\beta_1\rangle$, which has finite projective dimension. Since the modules $\langle\alpha\rangle, \langle\alpha^2\rangle, \ldots, \langle\alpha^t\rangle$ have S_x and S_y as the only composition factors, they are of finite projective dimension. Thus $P_x \supset \langle\alpha\rangle \supset \langle\alpha^2\rangle \supset \ldots \langle\alpha^t\rangle \supset 0$ is a suitable α-filtration.

□

Proposition 3.18
If $x^+ = \{\alpha, \beta_1, \gamma\}$, then there exists an α-filtration $\mathcal{F}$ of P_x having finite projective dimension.

Proof. By lemmata 3.16 and 3.17 we can assume that $\mathcal{L} \not\subseteq \{\alpha^t, \alpha^2\beta_1\}$ and $\mathcal{L} \not\subseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$. Then $\mathrm{pdim}_\Lambda\langle\alpha^k\rangle < \infty$ for $2 \leq k \leq t$ since $\langle\alpha^k\rangle$ has only S_x as a composition factor by 3.13 a). Moreover, $\mathrm{pdim}_\Lambda\langle\alpha, \beta_1, \gamma\rangle < \infty$ since it is the left hand term of the following exact sequence:

$$0 \to \langle\alpha, \beta_1, \gamma\rangle \to P_x \to S_x \to 0.$$

By Lemma 3.12 a) only the following two cases are possible:

i) $\alpha\beta_1 = 0$: Consider the following exact sequence:

$$0 \to \langle\beta_1, \gamma, \alpha^2, \alpha\gamma\rangle \to \langle\alpha, \beta_1, \gamma\rangle \to S_x \to 0.$$

Then $\mathrm{pdim}_\Lambda\langle\beta_1, \gamma, \alpha^2, \alpha\gamma\rangle < \infty$. By 3.15 c) we have $\langle\beta_1, \gamma, \alpha^2, \alpha\gamma\rangle = \langle\beta_1, \gamma, \alpha^2\rangle \oplus \langle\alpha\gamma\rangle$; hence $\mathrm{pdim}_\Lambda\langle\alpha\gamma\rangle < \infty$. Therefore $P_x \supset \langle\alpha, \beta_1, \gamma\rangle \supset \langle\alpha^2\rangle \oplus \langle\alpha\gamma\rangle \supset \langle\alpha^3\rangle \supset \ldots \langle\alpha^t\rangle \supset 0$ is a suitable α-filtration.

ii) $\alpha\gamma = 0$: Then $\mathrm{pdim}_\Lambda\langle\beta_1, \gamma, \alpha^2, \alpha\beta_1\rangle < \infty$ since we have the exact sequence

$$0 \to \langle\beta_1, \gamma, \alpha^2, \alpha\beta_1\rangle \to \langle\alpha, \beta_1, \gamma\rangle \to S_x \to 0.$$

If $\langle\gamma\rangle \cap \langle\alpha\beta_1\rangle = 0$, then by 3.15 a) we have $\langle\beta_1, \gamma, \alpha^2, \alpha\beta_1\rangle = \langle\beta_1, \gamma, \alpha^2\rangle \oplus \langle\alpha\beta_1\rangle$; hence $\mathrm{pdim}_\Lambda\langle\alpha\beta_1\rangle < \infty$. Therefore $P_x \supset \langle\alpha, \beta_1, \gamma\rangle \supset \langle\alpha^2\rangle \oplus \langle\alpha\beta_1\rangle \supset \langle\alpha^3\rangle \supset \ldots \langle\alpha^t\rangle \supset 0$ is a suitable α-filtration.
By Lemma 3.13 e) it remains to consider the case $\langle\gamma\rangle \cap \langle\beta_1\rangle = 0$: Then $\langle\beta_1, \gamma, \alpha^2, \alpha\beta_1\rangle = \langle\beta_1, \alpha^2\rangle \oplus \langle\gamma, \alpha\beta_1\rangle$ by 3.15 b). Thus $\mathrm{pdim}_\Lambda\langle\gamma, \alpha\beta_1\rangle < \infty$. Now $P_x \supset \langle\alpha, \beta_1, \gamma\rangle \supset \langle\alpha^2\rangle \oplus \langle\gamma, \alpha\beta_1\rangle \supset \langle\alpha^3\rangle \supset \ldots \langle\alpha^t\rangle \supset 0$ is a suitable α-filtration.

□

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